0J2 - Mechanics Lecture Notes 4

Dynamics II – Forces and motion in 2D and 3D

This is the generalisation of Dynamics I to 2D and 3D.

We shall use vectors and vector algebra including the scalar (dot) product.

1.1 General formulae for position, velocity and acceleration

Just as in 1D we start by considering just the motion. Forces are not involved explicitly.

We shall mainly use 3D to begin with. 2D is easily obtained from this by ignoring the z-coordinate.

Position is given by the position vector.

$$\begin{aligned} \mathbf{r} &= x\mathbf{i} + y\mathbf{j} \qquad \text{in 2D} \\ \mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad \text{in 3D} \end{aligned}$$

where (x, y) are the coordinates in 2D and (x, y, z) are the coordinates in 3D.

$\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the fundamental (Cartesian) unit vectors.

Since the coordinates can change in time we write

$$\begin{aligned} \mathbf{r}(t) &= x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad \text{(2D)} \\ \mathbf{r}(t) &= x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \qquad \text{(3D)} \end{aligned}$$

Velocity is the rate of change of position

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \qquad \text{(2D)}$$

or, using the 'dot' notation

$$\mathbf{v} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j}$$

where $\dot{x}$ means $\dfrac{dx}{dt}$ etc.

In 3D

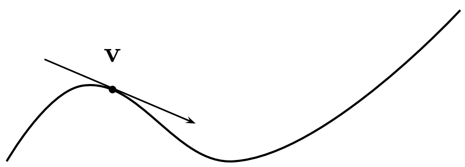
$$\mathbf{v} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} + \dot{z}\,\mathbf{k}$$

We often write

$$\mathbf{v} \;=\; v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$$

where $v_1 = \dot{x}$ is the x-component of $\mathbf{v}$
$v_2 = \dot{y}$ is the y-component of $\mathbf{v}$
$v_3 = \dot{z}$ is the z-component of $\mathbf{v}$

The direction of the velocity is always tangential to the path of the particle.



The <u>speed</u> is the magnitude (or modulus) of the velocity

$$v \;=\; |\mathbf{v}| \;=\; \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \qquad \text{in 3D}$$

<u>Acceleration</u> is rate of change of velocity

$$\mathbf{a} \;=\; \frac{d\mathbf{v}}{dt} \;=\; \frac{d^2\mathbf{r}}{dt^2} \;=\; \ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j} + \ddot{z}\,\mathbf{k} \qquad \text{in 3D}$$

where $\ddot{x} \;=\; \dfrac{d^2x}{dt^2}$ etc.

If we write $\mathbf{v} \;=\; v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ then

$$\mathbf{a} \;=\; \dot{v}_1\,\mathbf{i} + \dot{v}_2\,\mathbf{j} + \dot{v}_3\,\mathbf{k}$$

Note that acceleration and velocity may be in <u>different directions</u>.

Example 1 A particle moves in 2D. It is initially at the position $(1, 2)$.
The velocity is given by $\mathbf{v} \;=\; 6t^2\mathbf{i} + (1 - 2t)\mathbf{j}$.
Find the acceleration and the position at time t.

The acceleration is the differential of the velocity so

$$\mathbf{a} \;=\; \frac{d\mathbf{v}}{dt} \;=\; (12t)\,\mathbf{i} - 2\,\mathbf{j}$$

Now, writing the position vector as $\mathbf{r} \;=\; x\,\mathbf{i} + y\,\mathbf{j}$, the velocity is the differential of the position so

$$\mathbf{v} \;=\; \frac{d\mathbf{r}}{dt} \;=\; \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} \;=\; 6t^2\,\mathbf{i} + (1 - 2t)\,\mathbf{j}$$

Components must agree so

$$\frac{dx}{dt} = 6t^2 \qquad \text{and} \qquad \frac{dy}{dt} = 1 - 2t$$

Integrating these gives

$$x = 2t^3 + k_1 \qquad \text{and} \qquad y = t - t^2 + k_2$$

where k_1 and k_2 are the constants of integration.

At $t = 0$, $\quad x = 1$ and $y = 2$ so

$$1 = 0 + k_1 \qquad \text{and} \qquad 2 = 0 - 0 + k_2$$

Thus $k_1 = 1$ and $k_2 = 2$ so the position at time t is

$$\mathbf{r} = (2t^3 + 1)\,\mathbf{i} + (t - t^2 + 2)\,\mathbf{j}$$

or in coordinate form $(x, y) = (2t^3 + 1, t - t^2 + 2)$.

Summary of the relation between position $\mathbf{r}$, velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ for general motion in 2D or 3D using vectors:

$$\begin{array}{ccccc} & \text{differentiate} & & \text{differentiate} & \\ & \longrightarrow & & \longrightarrow & \\ \mathbf{r} & & \mathbf{v} & & \mathbf{a} \\ & \longleftarrow & & \longleftarrow & \\ & \text{integrate} & & \text{integrate} & \end{array}$$

When integrating we need extra information to determine the constants of integration. For example we might know the values of $\mathbf{r}$ and/or $\mathbf{v}$ at $t = 0$.

1.2 Constant acceleration

Just as in 1D the case of constant acceleration is important and we shall treat it in detail. We shall work in 2D but the generalisation to 3D will be simple.

Let $\mathbf{a} \;=\; a_1\,\mathbf{i} + a_2\,\mathbf{j}$.

a_1 and a_2 are the components of the acceleration and are constants.

Now $\mathbf{a} \;=\; \dot{v}_1\,\mathbf{i} + \dot{v}_2\,\mathbf{j}$ where v_1 and v_2 are the components of the velocity $\mathbf{v}$.

so $\dfrac{dv_1}{dt} \;=\; a_1$ and $\dfrac{dv_2}{dt} \;=\; a_2$.

Integrating

$$v_1 \;=\; a_1 t + u_1; \qquad\qquad v_2 \;=\; a_2 t + u_2$$

where u_1 and u_2 are constants of integration. Clearly at $t = 0$ $v_1 \;=\; u_1$ and $v_2 \;=\; u_2$ so u_1 is the values of v_1 and v_2 at $t = 0$.

Also $\mathbf{v} \;=\; \dfrac{d\mathbf{r}}{dt}$ and using $\mathbf{r} \;=\; x\,\mathbf{i} + y\,\mathbf{j}$ we have

$$\frac{dx}{dt} \;=\; v_1 \;=\; a_1 t + u_1 : \qquad\qquad \frac{dy}{dt} \;=\; v_2 \;=\; a_2 t + u_2$$

Integrating again

$$x \;=\; a_1\frac{t^2}{2} + u_1 t + c_1 : \qquad\qquad y \;=\; a_2\frac{t^2}{2} + u_2 t + c_2$$

where c_1 and c_2 are constants of integration. Clearly these are the values of x and y at $t = 0$.

$$\begin{aligned}
\text{Hence}\quad \mathbf{r} &= x\,\mathbf{i} + y\,\mathbf{j} \\
&= (c_1 + u_1 t + a_1\frac{t^2}{2})\,\mathbf{i} + (c_2 + u_2 t + a_2\frac{t^2}{2})\,\mathbf{j} \\
&= (c_1\,\mathbf{i} + c_2\,\mathbf{j}) + (u_1\,\mathbf{i} + u_2\,\mathbf{j})t + (a_1\,\mathbf{i} + a_2\,\mathbf{j})\frac{t^2}{2} \\
\mathbf{r} &= \mathbf{c} + \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2 \qquad\qquad (1)
\end{aligned}$$

$$\begin{aligned}
\text{where}\quad \mathbf{c} &= c_1\,\mathbf{i} + c_2\,\mathbf{j} \quad \text{is the initial position} \\
\mathbf{u} &= u_1\,\mathbf{i} + u_2\,\mathbf{j} \quad \text{is the initial velocity} \\
\mathbf{a} &= a_1\,\mathbf{i} + a_2\,\mathbf{j} \quad \text{is the constant acceleration}
\end{aligned}$$

All three of these are constant vectors.

Since $\mathbf{c}$ is the initial position, $\mathbf{r}-\mathbf{c}$ is the change in position, i.e. the displacement $\mathbf{s}$. Thus we can write (1) as

$$\mathbf{s} \;=\; \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2\,. \qquad (2)$$

Clearly this is a generalisation of the 1D formula $s \;=\; ut + \frac{1}{2}at^2$.

In 3D equations (1) and (2) are exactly the same except that the vectors now have three components instead of two.

Note 1 If we have motion in 3D with constant acceleration then all the motion takes place in a single plane. This is the plane containing the velocity vector at the start (or any other time) and the acceleration vector.

The motion is therefore effectively in 2D. e.g. a projectile.

The motion of a simple pendulum is also in a single plane, although the acceleration in not constant.

Note 2 Often we can treat the components separately as 1D problems so that a 2D or 3D problem becomes two or three separate 1D problems.

This is especially true for the constant acceleration case above.

1.3 Projectiles

An important application is the motion of things thrown or dropped which move under the action of gravity. These are called projectiles and the study of their motion is called ballistics.

The motion is in 3D but we usually choose the plane in which motion occurs to be the XY-plane so the motion is effectively in 2D.

In this plane we choose axes so the the Y-axis is up, and the X-axis is to the right.

The acceleration due to gravity is constant and is downwards so

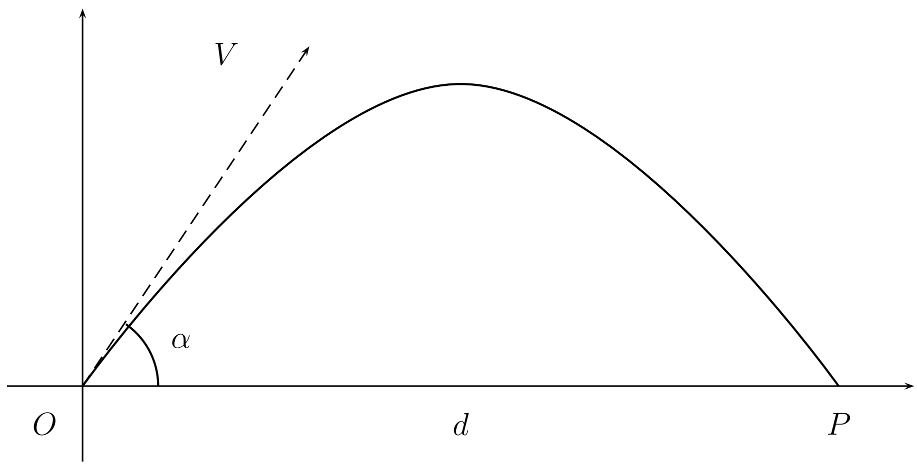
$$\mathbf{a} \;=\; -\,g\,\mathbf{j}$$

We shall discuss this by means of an example which has most of the important features.

Example:

A stone is thrown from the origin with speed V at an angle α above the horizontal. Assuming the ground is horizontal, how far does it travel before it lands?

Let the horizontal distance travelled be d, and let the time of flight be T.



$$\text{Initially} \quad \mathbf{r} = 0$$
$$\text{and} \quad \mathbf{v} = V\cos\alpha\,\mathbf{i} + V\sin\alpha\,\mathbf{j}$$

N.B. $\mathbf{r} = 0$ is an abbreviation for $\mathbf{r} = 0\,\mathbf{i} + 0\,\mathbf{j}$.

We shall do this problem by splitting the motion into horizontal and vertical parts. Each of these is 1D and we can use the usual 1D formulae for motion with constant acceleration.

$$s = ut + \frac{1}{2}at^2$$
$$s = \left(\frac{u+v}{2}\right)t$$
$$v = u + at$$
$$v^2 = u^2 + 2as$$

(We shall do the vector form afterwards).

Horizontally acceleration $a = 0$
initial velocity $u = V\cos\alpha$
final velocity $v = u + at = u = V\cos\alpha$
distance travelled $s = \left(\frac{u+v}{2}\right)t = ut$
and $s = d$ and $t = T$ so $d = (V\cos\alpha)T$ (1)

Vertically (upwards) acceleration $= -g$

initial velocity $u = V\sin\alpha$
distance travelled $s = 0$ since ir reaches the ground at P which is at same height as O.

Use the formula $s = ut + \frac{1}{2}at^2$ so

$$0 = (V\sin\alpha)T + \frac{1}{2}(-g)T^2 \qquad (2)$$

rearranging this gives

$$T\left(\frac{1}{2}gT - V\sin\alpha\right) = 0$$

so either $T = 0$ (this is the start) OR

$$T = \frac{2V}{g}\sin\alpha \qquad \text{(this is the end)}$$

Now use (1)

$$d = V\cos\alpha\left(\frac{2V}{g}\sin\alpha\right)$$
$$= \frac{2V^2}{g}\sin\alpha\cos\alpha = \frac{V^2}{g}\sin(2\alpha) \qquad \text{Ans.}$$

<u>Vector notation</u> The vector form of this is very similar. We use the vector formula

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

and put $\mathbf{s} = x\,\mathbf{i} + y\,\mathbf{j}$. The initial velocity is

$$\mathbf{u} = V\cos\alpha\,\mathbf{i} + V\sin\alpha\,\mathbf{j}$$

since the horizontal and vertical components of the velocity are $V\cos\alpha$ and $V\sin\alpha$. The acceleration is downwards so $\mathbf{a} = -g\,\mathbf{j}$. Thus

$$x\,\mathbf{i} + y\,\mathbf{j} = (V\cos\alpha\,\mathbf{i} + V\sin\alpha\,\mathbf{j})t + \frac{1}{2}(-g\,\mathbf{j})t^2 \qquad (3)$$

At time $t = T$, $x = d$ and $y = 0$ so

$$d\,\mathbf{i} = (V\cos\alpha\,\mathbf{i} + V\sin\alpha\,\mathbf{j})\,T - \frac{1}{2}g\,\mathbf{j}\,T^2\,.$$

Comparing the $\mathbf{i}$ coefficients on both sides gives

$$d = V\cos\alpha\,T \qquad (1),$$

which is the same as the previous (1).

Comparing the $\jmath$ coefficients on both sides gives

$$0 = V \sin\alpha\, T - \frac{1}{2} g\, T^2 \qquad (2)$$

which is the same as the previous (2). Since (1) and (2) are the same as before, the solution is the same i.e.

$$d = \frac{V^2}{g} \sin(2\alpha)\,.$$

In practice it is usually simpler to deal with the horizontal and vertical motion separately for projectiles.

Other points from this example

1. The horizontal distance travelled d is called the range.

2. The time of flight is $T = \dfrac{2V}{g} \sin\alpha$.

3. We can calculate the greatest height h. At the highest point the vertical velocity is zero. Using $v^2 = u^2 + 2as$ gives

$$0 = (V \sin\alpha)^2 - 2gh$$

$$\text{so} \quad h = \frac{V^2}{2g} \sin^2\alpha\,.$$

4. The time to reach the greatest height is $\frac{1}{2}T$. (left as an exercise.)

5. Angle α for maximum range. We have seen that, in general, the range is given by

$$d = \frac{V^2}{g} \sin(2\alpha)\,.$$

For a given V this is largest when $\sin(2\alpha) = 1$. Therefore $2\alpha = 90^\circ$ and thus

$$\alpha = 45^\circ\,.$$

The actual maximum range is $\dfrac{V^2}{g}$.

6. The equation of the path. From (3) we have

$$x = V \cos\alpha\, t$$

$$\text{and} \quad y = V \sin\alpha\, t - \frac{1}{2} g\, t^2$$

$$\text{so} \quad t = \frac{x}{V\cos\alpha}$$

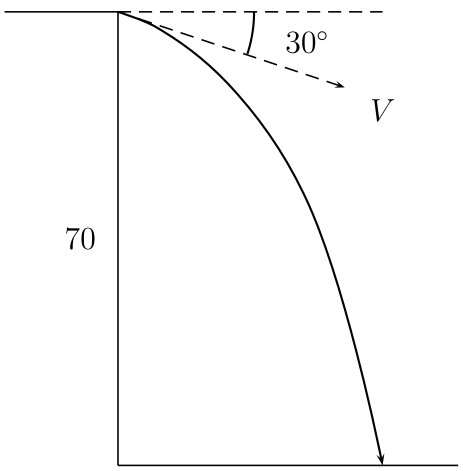
$$\begin{aligned}\text{and} \quad y &= (V\sin\alpha)\frac{x}{V\cos\alpha} - \frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2 \\ &= (\tan\alpha)x - \frac{g}{2V^2\cos^2\alpha}x^2 \\ &= Ax + Bx^2 \qquad (4)\end{aligned}$$

where

$$A = \tan\alpha \qquad \text{and} \qquad B = -\frac{g}{2V^2}\sec^2\alpha\,.$$

Equation (4) is the equation of a parabola.

Example A stone is thrown from the top of a cliff of height 70 m at an angle of 30° below the horizontal. It hits the sea 20 m from the base of the cliff. Find the initial speed of the stone and also the time of flight. Take $g = 9.81 \text{ ms}^{-2}$.



Let the initial speed be V. Let the time of flight be T.

Horizontally $a = 0$. $u = V\cos 30° = V\frac{\sqrt{3}}{2}$. $s = 20$.

Using $s = ut + \frac{1}{2}at^2$ gives

$$20 = V\frac{\sqrt{3}}{2}T + 0 \quad \Rightarrow \quad T = \frac{40}{\sqrt{3}V} \qquad (1)$$

Vertically (downwards) $\quad a = g. \quad u = V\sin 30° = \frac{1}{2}V. \quad s = 70.$

$$70 = (\frac{1}{2}V)T + \frac{1}{2}gT^2 \qquad (2)$$

Using (1) for T this becomes

$$70 = \frac{1}{2}V\left(\frac{40}{\sqrt{3}V}\right) + \frac{1}{2}g\left(\frac{40}{\sqrt{3}V}\right)^2$$

$$70 = \frac{20}{\sqrt{3}} + \frac{800 \times 9.81}{3V^2}$$

$$\frac{7848}{3V^2} = 70 - \frac{20}{\sqrt{3}} = 58.45$$

$$V^2 = \frac{7848}{3 \times 58.45} = 44.76$$

$$\therefore \quad V = 6.69\,\mathrm{ms}^{-1}$$

Also from (1)

$$T = \frac{40}{\sqrt{3} \times 6.69} = 3.452 \text{ seconds.}$$

1.4 Force, Momentum and Impulse in 2D and 3D

For a particle of mass m, Newton's Second Law (N2) in vector form is

$$\boxed{\mathbf{F} = m\,\mathbf{a}} \qquad (1)$$

so the direction of the acceleration is the same as the direction of the force. Otherwise the law is the same as in 1D.

Similarly momentum in vector form is $m\,\mathbf{v}$.

The rate of change of momentum, assuming mass m is constant is

$$\frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} = \mathbf{F} \qquad (2)$$

so if $\mathbf{F} = 0$ then momentum is conserved.

Kinetic energy is given by

$$\mathrm{KE} = \frac{1}{2}m\mathbf{v}.\mathbf{v} = \frac{1}{2}mv^2 \qquad (3)$$

where v is the magnitude of vector $\mathbf{v}$, which is the same as in 1D.

Note the use of the scalar product of a vector with itself, which gives the magnitude of the vector:

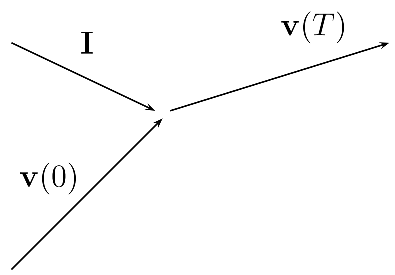
$$\mathbf{v}.\mathbf{v} \;=\; (v_x, v_y, v_z).(v_x, v_y, v_z) \;=\; v_x^2 + v_y^2 + v_z^2 \;=\; v^2$$

Impulse is now a vector

$$\begin{aligned}
\mathbf{I} &= \int_0^T \mathbf{F}\, dt \\
&= \int_0^T m\mathbf{a}\, dt \\
&= m \int_0^T \frac{d\mathbf{v}}{dt}\, dt \\
&= m[\mathbf{v}(T) - \mathbf{v}(0)] \qquad (4)
\end{aligned}$$

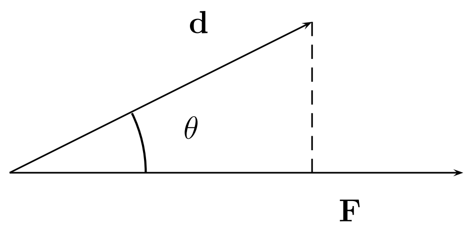
Hence $\mathbf{I}$ gives the change in momentum, just as in 1D.

However, the direction of $\mathbf{I}$ is not in general the same as that of $\mathbf{v}$,



$\mathbf{v}(0)$, $\mathbf{v}(T)$ and $\mathbf{I}$ can be in three different directions!

Work done is force × distance moved in the direction of the force.



For diplacement $\mathbf{d}$ and force $\mathbf{F}$ we have $W = Fd\cos\theta$ or

$$\boxed{W = \mathbf{F}.\mathbf{d}} \qquad (5)$$

Note This formula is true for constant forces. If the force is not constant then it is usually better to consider changes in KE instead.

Power is now defined as

$$P = \mathbf{F}.\mathbf{v}$$

and for constant forces

$$\frac{dW}{dt} = \mathbf{F}.\frac{d}{dt}(\mathbf{d}) = \mathbf{F}.\mathbf{v} = P$$

so the power is the rate of doing work, as in 1D.

Example A force $\mathbf{F} = 4t\,\mathbf{i} + 2\sin t\,\mathbf{j}$ acts on a particle of mass 2 kg which is initially at $\mathbf{r} = 0$ with $\mathbf{v} = 0$. Find

(i) the velocity at a function of time.

(ii) the work done by the force between $t = 0$ and $t = T$.

(iii) the power of $\mathbf{F}$ at time T.

Answer (i) By N2 $\mathbf{F} = m\mathbf{a}$

$$\therefore \quad \mathbf{a} = \frac{1}{2}\mathbf{F} = 2t\,\mathbf{i} + \sin t\,\mathbf{j}$$

and since $\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$

$$\frac{d\mathbf{v}}{dt} = 2t\,\mathbf{i} + \sin t\,\mathbf{j}$$

Integrating

$$\mathbf{v} = (t^2 + c_1)\,\mathbf{i} + (-\cos t + c_2)\,\mathbf{j}$$

where c_1 and c_2 are constants of integration.
At $t = 0$

$$\mathbf{v} = c_1\,\mathbf{i} + (-1 + c_2)\,\mathbf{j} = 0\,\mathbf{i} + 0\,\mathbf{j}$$

Thus $c_1 = 0$ and $c_2 = 1$ and so

$$\mathbf{v} = t^2\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}$$

(ii) By conservation of energy, the work done by the force must equal the change in KE.

at $t = 0$, $\mathbf{v} \;=\; 0$ so $v = 0$ and KE $\;=\; \frac{1}{2}mv^2 \;=\; 0$.

at $t = T$, $\mathbf{v} \;=\; T^2\,\mathbf{i} + (1 - \cos T)\,\mathbf{j}$ so

$$v^2 \;=\; \mathbf{v}.\mathbf{v} \;=\; T^4 + (1 - \cos T)^2$$

$$\text{KE} \;=\; \frac{1}{2}mv^2 \;=\; \frac{1}{2} \times 2 \times v^2 \;=\; v^2 \;=\; T^4 + (1 - \cos T)^2$$

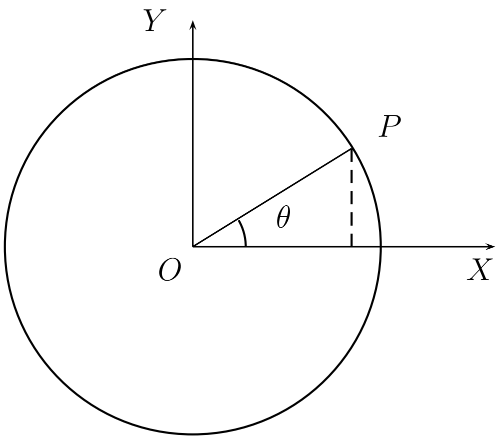
and this is the work done.

(iii)

$$\begin{aligned} \text{Power} \quad P &= \mathbf{F}.\mathbf{v} \\ &= (4T\,\mathbf{i} + 2\sin T\,\mathbf{j}).(T^2\,\mathbf{i} + (1 - \cos T)\,\mathbf{j}) \\ &= 4T^3 + 2\sin T(1 - \cos T) \end{aligned}$$

1.5 Motion in a circle at constant speed

Suppose a particle moves in 2D on a circle of radius R centred at O.



When the angle of the position vector $\vec{OP}$ is θ to the X-axis the coordinates are $(R\cos\theta,\ R\sin\theta)$ the position vector is

$$\mathbf{r} \;=\; R\cos\theta\,\mathbf{i} + R\sin\theta\,\mathbf{j}$$

Since θ is the angle we say that the rate of change of θ, $\dfrac{d\theta}{dt}$ is the angular velocity.

The units are radians/sec (for planets it could be radians per year!).

We shall assume that the angular velocity is constant. Let this be ω (Greek letter omega).

$$\text{so}\quad \frac{d\theta}{dt} \;=\; \omega \qquad \text{(constant)}$$

Integrating gives

$$\theta \;=\; \omega\, t + \phi$$

where ϕ (Greek letter phi) is the constant of integration, sometimes called the phase.

We choose θ to be 0 at $t = 0$ so $\phi \;=\; 0$.

We now have

$$\mathbf{r} \;=\; R\cos(\omega t)\,\mathbf{i} + R\sin(\omega t)\,\mathbf{j} \qquad (1)$$

Differentiating gives the velocity

$$\mathbf{v} \;=\; -\,\omega R\sin(\omega t)\,\mathbf{i} + \omega R\cos(\omega t)\,\mathbf{j} \qquad (2)$$

The speed v is the magnitude of the velocity vector so

$$\begin{aligned} v^2 &= [-\omega R \sin(\omega t)]^2 + [\omega R \cos(\omega t)]^2 \\ &= \omega^2 R^2 \sin^2(\omega t) + \omega^2 R^2 \cos^2(\omega t) \\ &= \omega^2 R^2 [\sin^2(\omega t) + \cos^2(\omega t)] \\ &= \omega^2 R^2 \end{aligned}$$

Taking the square root $\boxed{v = \omega R}$ (3).
Clearly v is constant so for circular motion, constant angular velocity means constant speed and vice versa.

The period is the time taken for one complete revolution. We use the symbol τ (Greek letter tau) for this. The distance is $2\pi R$ and the speed is ωR so the time taken is $\dfrac{2\pi R}{\omega R}$ and so

$$\text{Period} \quad \tau = \frac{2\pi}{\omega} \qquad (4)$$

Aside If $\theta \neq 0$ at $t = 0$ then $\phi \neq 0$ so

$$\mathbf{r} = R\cos(\omega t + \phi)\,\mathbf{i} + R\sin(\omega t + \phi)\,\mathbf{j}$$

and

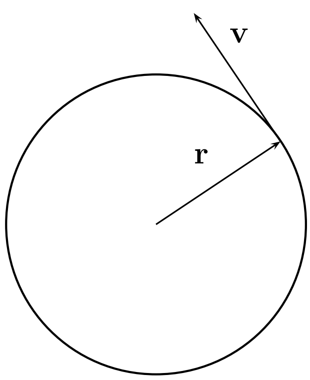
$$\mathbf{v} = -\omega R\sin(\omega t + \phi)\,\mathbf{i} + \omega R\cos(\omega t + \phi)\,\mathbf{j}$$

but again we get $v^2 = \omega^2 R^2$. (Left as an exercise). (end of aside).

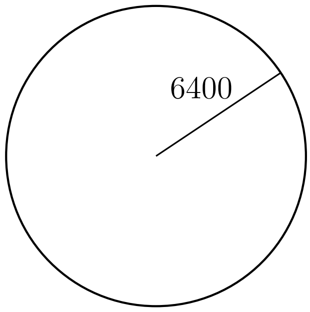
We expect the direction of the velocity $\mathbf{v}$ to be the tangent to the circle, i.e. perpendicular to the radius. We can prove this by calculating

$$\begin{aligned} \mathbf{r}.\mathbf{v} &= [R\cos(\omega t)\,\mathbf{i} + R\sin(\omega t)\,\mathbf{j}].[-\omega R\sin(\omega t)\,\mathbf{i} + \omega R\cos(\omega t)\,\mathbf{j}] \\ &= -\omega R^2\cos(\omega t)\sin(\omega t) + \omega R^2\cos(\omega t)\sin(\omega t) \\ &= 0 \end{aligned}$$

so $\mathbf{v}$ is perpendicular to $\mathbf{r}$ and so the direction of $\mathbf{v}$ is the tangent to the circle



Example A student sits on the equator. If the radius of the earth is 6400 km what is the speed of the student in km/hr (relative to the centre of the earth)?



Answer The earth does one revolution of 2π radians in 24 hours.

Therefore the angular velocity is

$$\omega = \frac{2\pi}{24} = \frac{\pi}{12} \quad \text{radians per hour}$$

$$\begin{aligned} \text{speed} \quad v = \omega R &= \frac{\pi}{12} \times 6400 \text{ km/hr} \\ &= \frac{1600\pi}{3} \\ &= 1676 \text{ km/hr} \end{aligned}$$

(end of example)

We now have

$$\begin{aligned} \mathbf{r} &= R\cos(\omega t)\,\mathbf{i} + R\sin(\omega t)\,\mathbf{j} \qquad (1) \\ \mathbf{v} &= -\,\omega R\sin(\omega t)\,\mathbf{i} + \omega R\cos(\omega t)\,\mathbf{j} \qquad (2) \\ \text{so} \quad \mathbf{a} &= \frac{d\mathbf{v}}{dt} = -\,\omega^2 R\cos(\omega t)\,\mathbf{i} - \omega^2 R\sin(\omega t)\,\mathbf{j} \\ &= -\,\omega^2[R\cos(\omega t)\,\mathbf{i} + R\sin(\omega t)\,\mathbf{j}] \\ &= -\,\omega^2\mathbf{r} \qquad (5). \end{aligned}$$

This implies that the acceleration is directed towards the centre of the circle.

The force needed is

$$\mathbf{F} = m\mathbf{a} = -\,m\omega^2\mathbf{r} \qquad (6)$$

so the force is also directed towards the centre of the circle. It is called the centripetal force. Sometimes it is wrongly called the centrifugal force which would be a force away from the centre.

The magnitude of the acceleration is given by

$$a \;=\; |\mathbf{a}| \;=\; |-\omega^2\mathbf{r}| \;=\; \omega^2 R \qquad (7a)$$

since $|\mathbf{r}|$ is always R wherever it is on the circle.

Since $v \;=\; \omega R$ we can also write

$$a \;=\; \omega v \qquad (7b)$$

$$\text{or} \quad a \;=\; \frac{v^2}{R} \qquad (7c)$$

Example Find the magnitude of the acceleration of the student in the previous example.

We have $\omega \;=\; \dfrac{\pi}{12}$ km/hr, and $R \;=\; 6400km$.

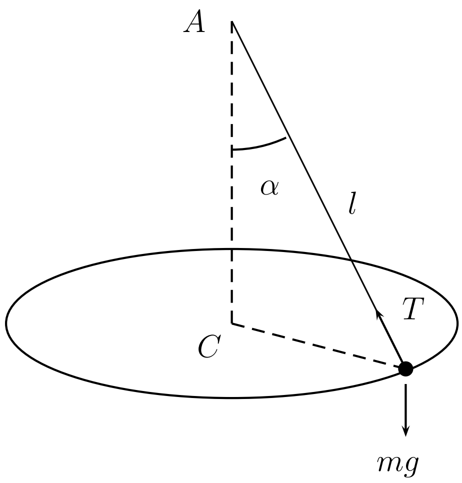
$$\begin{aligned} \text{so} \quad a \;&=\; \omega^2 R \\ &=\; \frac{\pi^2}{144} \times 6400 \quad \text{km/hr}^2 \\ &=\; 438.6 \quad \text{km/hr}^2 \end{aligned}$$

(we could convert to ms^{-2} by multiplying by $\dfrac{1000}{3600^2}$.)

An application of circular motion at constant speed is the

Conical pendulum

This consists of a light inelastic string attached at one end to a fixed point A. The other end is attached to a particle of mass m which is moving in a horizontal circle whose centre O is directly below A.



Let the length of the string be l.
Let the angle with the vertical be α.
Let the tension in the string be T.
Let the speed be v.

The radius of the circle is $R = l \sin\alpha$.

so the angular velocity $\omega = \dfrac{v}{R} = \dfrac{v}{l\sin\alpha}$.

Here the tension T does two jobs. The vertical component balances the weight mg so that there is no vertical motion.

The horizontal component acts towards C and provides the centripetal force necessary for the circular motion.

Vertically

$$T\cos\alpha = mg \qquad \text{so} \qquad T = \frac{mg}{\cos\alpha}.$$

Clearly we could never have $\alpha = 90°$ which would require T to be infinite!

Horizontally, towards the centre C

$$\text{force} = T\sin\alpha \quad \text{so the acceleration is} \quad a = \frac{T\sin\alpha}{m} .$$

But $a = \omega^2 R = \omega^2 l \sin\alpha$.

$$\therefore \quad \omega^2 l \sin\alpha = \frac{T\sin\alpha}{m}$$
$$\omega^2 = \frac{T}{ml}$$
$$= \frac{1}{ml}\frac{mg}{\cos\alpha}$$
$$\therefore \quad \boxed{\omega^2 = \frac{g}{l\cos\alpha}}.$$

Notice that this result is independent of m!

This gives the relation between ω, l and α for the conical pendulum.

We can also calculate

$$v = \omega R = l\sin\alpha\sqrt{\frac{g}{l\cos\alpha}}$$

and the period $\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l\cos\alpha}{g}}$

Example A conical pendulum has string of length 2 m. If the pendulum makes 1 revolution per second, find the angle the string makes with the vertical.

Solution

One revolution $= 2\pi$ radians.

Therefore the angular velocity $\omega = 2\pi$ radians/sec.

Let the angle the string makes with the vertical be α.

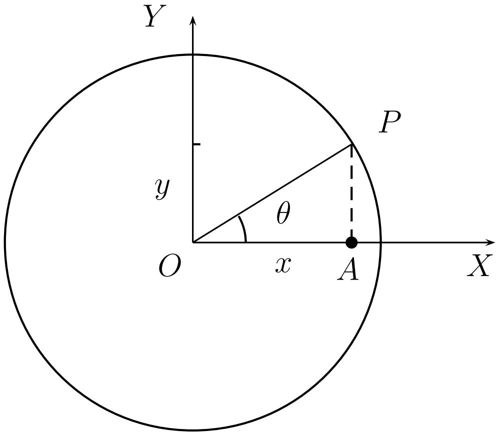
From the above

$$\omega^2 = \frac{g}{l\cos\alpha}$$
$$\therefore \quad \cos\alpha = \frac{g}{\omega^2 l} = \frac{9.81}{(2\pi)^2 \times 2} = 0.1242$$
$$\therefore \quad \alpha = 1.4462 \text{ radians} = 82.86^\circ.$$

where we have taken $g = 9.81 \text{ ms}^{-2}$.

Note that this particle is going rather fast. This results in a high value for α, so the string is getting close to horizontal.

Final note on circular motion at constant speed



The position vector is $\mathbf{r} \;=\; x\,\mathbf{i} + y\,\mathbf{j}$ where $x \;=\; R\cos\theta$ and $y \;=\; R\sin\theta$.

As before $\theta \;=\; \omega t$ choosing the phase to be zero.

We shall consider only the X-coordinate, i.e.the motion of the point A along the X-axis as the point P travels around the circle.

$$x \;=\; R\cos(\omega t)$$

so $$\frac{dx}{dt} \;=\; \dot{x} \;=\; -\,\omega R\sin(\omega t)$$

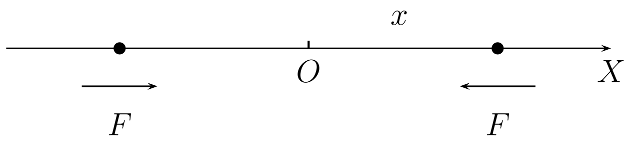
and $$\frac{d^2x}{dt^2} \;=\; \ddot{x} \;=\; -\,\omega^2 R\cos(\omega t)$$

so clearly $\boxed{\ddot{x} \;=\; -\,\omega^2 x.}$

This is the equation of simple harmonic motion (SHM) in 1D.

1.6 Simple Harmonic Motion in 1D

A particle of mass m moves in 1D along the X-axis and has position x at time t. Suppose the force acting upon it is $F = -kx$, where k is a positive constant. Clearly F always acts towards the origin.



As usual $ma = F$
so $m\ddot{x} = -kx$
and $\ddot{x} = -\dfrac{k}{m}x$

Now k is positive and so is m so put $\dfrac{k}{m} = \omega^2$ to get

$$\boxed{\ddot{x} = -\omega^2 x} \qquad (1)$$

as we had for the x-component of circular motion.

(1) is the equation for <u>Simple Harmonic Motion</u> (SHM)

It is easy to verify that $x = R\cos(\omega t + \phi)$ is the solution, where R (the amplitude) and ϕ (the phase) are constants. Sometimes we can choose ϕ to be zero so in this case $x = R\cos\omega t$.

This motion occurs, for example, when a particle moves up and down on an elastic string or spring. It also describes the motion of a simple pendulum provided the angle of swing is small. It occurs in <u>many</u> other engineering and science situations.

The value of ω is the given constant in (1) and is characteristic of the system.

The other constants R and ϕ are determined using extra information.
Often this is the starting situation, called the <u>initial conditions</u>.